

# Holography for degenerate boundaries

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## Abstract

We discuss the AdS/CFT correspondence for negative curvature Einstein manifolds whose conformal boundary is degenerate in the sense that it is of codimension greater than one. In such manifolds, hypersurfaces of constant radius do not blow up uniformly as one increases the radius; examples include products of hyperbolic spaces and the Bergman metric. We find that there is a well-defined correspondence between the IR regulated bulk theory and conformal field theory defined in a background whose degenerate geometry is regulated by the same parameter. We are hence able to make sense of supergravity in backgrounds such as  $AdS_3 \times H^2$ .

## I. INTRODUCTION

Following the now famous conjecture of Maldacena [1], [2], [3], [4] relating supergravity on anti-de Sitter spacetimes to a conformal field theory in one less dimension there has been a great deal of interest in supergravity compactifications on anti-de Sitter spaces. One of the most interesting features of the AdS/CFT correspondence is that it provides an example of the holographic principle [5]. In the context of the Maldacena conjecture, holography was first discussed in detail in [6] whilst discussions of holography in cosmology have appeared in [7], [8].

However, the holographic description of the anti-de Sitter bulk theory relies heavily on special features of the boundary, namely that the conformal boundary in the sense of Penrose [9] is a non-degenerate manifold of codimension one. Although most of the physical negative curvature solutions that one is interested in considering, such as black holes, satisfy these properties, several classes of negatively curved spacetimes do not.

In particular, coset spaces such as  $SO(3, 1)/SO(3)$ , Bergman type metrics and products of hyperbolic spaces have conformal boundaries of codimension greater than one, which we will refer to as degenerate boundaries. Typically, if  $1/\epsilon$  characterizes the radius of the  $(d + 1)$ -dimensional spacetime, then the volume of a hypersurface of constant  $\epsilon$  behaves as  $\epsilon^\alpha$  with  $\alpha > -d$ ; the induced hypersurface does not blow up uniformly as  $\epsilon \rightarrow 0$ . First steps

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towards understanding the bulk boundary correspondence in these cases were taken in [10] and the purpose of this paper is to take this correspondence further and deal with some of the unresolved issues of [10].

In particular, although the authors of [10] were able to analyse the correspondence between scalar fields in the bulk and boundary scalar operators for the Bergman metric, they did not find such an interpretation for coset spaces which we shall do here. Furthermore, to properly formulate the bulk/boundary correspondence, one needs to understand how the bulk partition function gives rise to the partition function for the boundary conformal field theory.

When the conformal boundary is non-degenerate, there is a well understood way of defining the bulk (Euclidean) action without introducing a background [11], [12], [13]. One introduces local boundary counterterms into the action, which remove all divergences and leave a finite action corresponding precisely to the partition function of the conformal field theory.

This procedure must break down when the boundary is degenerate, since there is no conformal frame in which the hypersurface radius  $\epsilon$  does not appear in the metric. This means that one cannot hope to eliminate all radius dependence from the partition function; put differently, as one takes the limit  $\epsilon \rightarrow 0$ , the partition function for the conformal field theory must diverge, since the geometry is becoming highly degenerate.

So the question is whether one can make sense of a correspondence between bulk and boundary partition functions. We will show that one can, provided that one takes the correspondence to be at finite radius. That is, an IR cutoff in the bulk theory will appear as a regulation of the conformal field theory target geometry. This is a novel manifestation of the IR/UV correspondence [6].

In several recent papers the relationship between the Randall/Sundrum scenario [14] and AdS/CFT correspondence has been explored [15], [16], [17]. In this context, one can view the five-dimensional negative curvature spacetime to be a construction from a symmetric non-degenerate four-dimensional brane world. In [14], [15], the brane world lives at finite radius  $R$ , and the induced action on the brane includes Einstein gravity plus corrections. For our “degenerate” brane worlds, however, the induced action on the brane does not include Einstein gravity, even when the brane is four-dimensional.

The plan of this paper is as follows. In §II, we discuss classes of metrics which have degenerate conformal boundaries. In §III, we discuss the interpretation of the gravitational bulk action for such spacetimes in terms of a dual conformal field theory. In §IV, we analyse how correlation functions for scalar operators in the conformal field theory may be derived from bulk massive scalar fields.

## II. CLASSES OF METRICS WITH DEGENERATE BOUNDARIES

Suppose that  $M$  is a complete Einstein manifold of negative curvature and dimension  $d + 1$  which has a conformal boundary which a  $d$ -manifold  $N$ . This means that the metric of  $M$  can be written near the boundary as

$$ds^2 = \frac{du^2}{u^2} + \frac{1}{u^2} g_{ij}(u, x) dx^i dx^j, \quad (2.1)$$

where  $u$  is a smooth function with a first order zero on  $N$  which is positive on  $M$ . Usually we assume that  $g_0 = g(0, x)$  is a non-degenerate metric on  $N$ , independently of how we take the limit  $u \rightarrow 0$ . However, more general negative curvature manifolds can have conformal boundaries which are degenerate in the sense that  $g(\epsilon, x)$  is divergent as  $\epsilon \rightarrow 0$ . The vielbeins for such a metric will not all be finite as  $\epsilon \rightarrow 0$ ; at least one will tend to zero or diverge.

Let us refer to  $N_\epsilon$  as the regulated boundary; then  $N_\epsilon$  will have a natural conformal structure. We will show in the following section that  $N_\epsilon$  must have negative curvature of the order of the  $(d+1)$ -dimensional cosmological constant when  $N$  is degenerate. We expect there to be a correspondence between conformal field theory on  $N_\epsilon$  with a UV cutoff  $1/\epsilon$  and quantum gravity on  $M$  with an IR cutoff  $\epsilon$ . To investigate this correspondence we will as usual consider the relation between bulk and boundary partition functions. Before we do so, however, it will be helpful to give several examples of spaces which have degenerate boundaries; we will be using these explicit examples to illustrate general arguments in the following sections.

We will consider here two generic classes of negative curvature manifolds which have degenerate conformal boundaries. Let us normalise the  $(d+1)$ -dimensional Einstein action such that

$$I_{\text{bulk}} = -\frac{1}{16\pi G_{d+1}} \int_M d^{d+1}x \sqrt{g} (R_g + d(d-1)l^2) - \frac{1}{8\pi G_{d+1}} \int_N d^d x \sqrt{\gamma} K, \quad (2.2)$$

where  $R_g$  is the Ricci scalar and  $K$  is the trace of the extrinsic curvature of the boundary  $N$  embedded in  $M$ .

An example of the first class of degenerate boundary manifolds was considered in [10]; it is included in the family of Einstein-Kähler metrics of the form

$$ds^2 = (l^2 r^2 + \frac{1}{4} - \frac{k}{r^4})^{-1} dr^2 + r^2 (l^2 r^2 + \frac{1}{4} - \frac{k}{r^4}) (d\psi + 2 \cos \theta d\phi)^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.3)$$

which was constructed in [18]. Regularity requires that the periodicity of  $\psi$  is

$$\beta = \frac{8\pi}{k} = \frac{4\pi}{U'(r_+)r_+}, \quad (2.4)$$

where the horizon location is  $r_+$  and  $U(r)$  is the metric function. The boundary admits the conformal structure

$$ds^2 = (l^2 r^2 + \frac{1}{4} - \frac{k}{r^4}) (d\psi + 2 \cos \theta d\phi)^2 + (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.5)$$

which is a squashed three-sphere of the form

$$ds^2 = l_3^2 (d\tilde{\psi} + \cos \theta d\phi)^2 + (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.6)$$

and becomes degenerate as we take the limit  $r \rightarrow \infty$ . The  $k=0$  metric is the Bergman metric which is the group manifold  $SU(2,1)$  discussed in [10]; it will be convenient to use here an alternative form of the Bergman metric

$$ds^2 = \frac{2}{l^2} [d\rho^2 + \frac{1}{4} \sinh^2 \rho \cosh^2 \rho (d\psi + \cos \theta d\phi)^2 + \frac{1}{4} \sinh^2 \rho (d\theta^2 + \sin^2 \theta d\phi^2)], \quad (2.7)$$

where the prefactor ensures that the curvature behaves as  $R_g = -12l^2$ . Although the boundary of the Bergman metric becomes degenerate in the infinite limit, there is a very natural choice of conformal boundary since this space, like others in the same family, is constructed by radially extending a  $U(1)$  bundle over an Einstein space.

Much that one says about the Bergman metric can also be extended to other radial extensions of bundles over compact spaces. Such spaces have been considered in the past in the context of compactifications [19] and a number of other examples are known. For example, in eight dimensions one could consider an Einstein hyper-Kähler manifold of the form

$$ds^2 = d\rho^2 + \frac{1}{4} \sinh^2 \rho [d\theta^2 + \frac{1}{4} \sin^2 \theta \omega_i^2 + \frac{1}{4} \cosh^2 \rho (\nu_i + \cos \theta \omega_i)^2], \quad (2.8)$$

where  $\nu_i, \omega_i$  are left-invariant forms satisfying  $SU(2)$  algebras. This metric is obtained by analytically continuing the standard Fubini-Study Einstein metric on  $HP^2$ , the quaternionic projective plane. Hypersurfaces of constant  $\rho$  are squashed seven-spheres which become infinitely squashed as one takes the boundary to infinity. Analysis of the AdS/CFT correspondence in this case would be very similar to that for the Bergman metric. One would also expect that one could find an Einstein metric which is a radial extension of a twisted  $S^2$  bundle over  $S^2$ . Hypersurfaces of constant radius should correspond to off-shell extensions of the Page metric [20].

We should mention that these metrics do not have a nice physical Lorentzian interpretation. The existence of a topologically non-trivial degenerate conformal boundary is related to the non-trivial fibration of  $U(1)$  coordinates which would have to be analytically continued to give a Lorentzian metric. Analytically continuing the Bergman metric leads to a Lorentzian metric which is complex and whose physical interpretation is unclear. Furthermore, not only are these metrics not supersymmetric (the Bergman metric is certainly not supersymmetric since there is no supergroup which has  $SU(2, 1)$  as its bosonic part) but also many such metrics will not even admit a spin structure.

Given the problems with the Lorentzian interpretation, we shouldn't be surprised if the thermodynamic quantities in the conformal field theory take unphysical values. It is well known [21] that a generic quantum field theory within a causality violating background will exhibit pathologies such as negative entropy which one interprets as reflecting the unphysical nature of the background and we will see similar phenomena here. However, although such backgrounds are not of great physical interest in string theory they provide interesting examples of a more general AdS/CFT correspondence. There may be other backgrounds in string theory which are physically interesting and exhibit similar degenerate behaviour.

The second category of manifolds in which we are interested is characterized by the existence of more than one "radial" coordinate. The simplest example which we will consider in the most detail is the product of hyperbolic manifolds of dimensions  $d_1$  and  $d_2$  respectively:

$$ds^2 = \frac{(d_1 - 1)}{(d_1 + d_2 - 1)l^2 z_1^2} (dz_1^2 + d\mathbf{x}_1^2) + \frac{(d_2 - 1)}{(d_1 + d_2 - 1)l^2 z_2^2} (dz_2^2 + d\mathbf{x}_2^2). \quad (2.9)$$

“Infinity” in this metric is obtained by taking either  $z_1$  or  $z_2$  to zero; however, for a holographic principle to be formulated we need to define a boundary of codimension one. One way to do this would be to effectively divide the boundary into two parts: consider a hypersurface of constant  $z_1 = \epsilon_1 \ll 1$  and a hypersurface of constant  $z_2 = \epsilon_2 \ll 1$  which are glued together along the hypersurface of codimension two  $z_1 = \epsilon_1; z_2 = \epsilon_2$ . However, since one would need to be careful about boundary conditions for fields along this join it is easier to work with the conformally equivalent surface defined by setting

$$z_1 = u \cos \theta, z_2 = u \sin \theta, \quad (2.10)$$

so that for example when  $d_1 = d_2 = 2$  the metric becomes

$$ds^2 = \frac{1}{3l^2} \left\{ \frac{2du^2}{u^2} + \frac{2dud\theta}{u} \left( \frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} \right) + \frac{(\sin^4 \theta + \cos^4 \theta)}{\sin^2 \theta \cos^2 \theta} d\theta^2 \right. \\ \left. + \frac{dx_1^2}{u^2 \cos^2 \theta} + \frac{dx_2^2}{u^2 \sin^2 \theta} \right\}. \quad (2.11)$$

With this choice of coordinates the induced metric on constant  $u$  hypersurfaces is

$$ds^2 = \frac{d\theta^2}{2 \sin^2 \theta \cos^2 \theta} + \frac{dx_1^2}{u^2 \cos^2 \theta} + \frac{dx_2^2}{u^2 \sin^2 \theta}, \quad (2.12)$$

which is degenerate as  $u \rightarrow 0$  and further degenerates when  $\theta \rightarrow 0, \pi/2$ . Notice that the metric is non-singular in this conformal frame, with the limits in  $\theta$  corresponding to non-singular tubes in the Euclidean metric.

As well as products of negative curvature manifolds, various coset spaces exhibit a similar degenerate behaviour. In particular, one can analytically continue manifolds which were considered in the context of supergravity compactifications, reviewed in [19]. The coset space  $SO(3,1)/SO(2)$  possesses an Einstein metric

$$ds^2 = \frac{1}{9} (d\psi + i \cosh \rho_1 d\phi_1 + i \cosh \rho_2 d\phi_2)^2 + \frac{1}{6} \sum_i (d\rho_i^2 + \sinh^2 \rho_i d\phi_i^2), \quad (2.13)$$

where we have taken  $l^2 = \frac{1}{4}$ . The Euclidean metric is not real, but one can find a Lorentzian section (discussed in [10]) which is; however, our Lorentzian theory will still exhibit pathologies since there are closed timelike curves.

The most useful choice of boundary in this case is probably to take

$$e^{\rho_1} = \frac{1}{u \cos \theta}; \quad e^{\rho_2} = \frac{1}{u \sin \theta}, \quad (2.14)$$

in analogy to the above. Then as  $u \rightarrow 0$  the leading order induced metric on a constant  $u$  surface is

$$ds^2 = \frac{d\theta^2}{12 \sin^2 \theta \cos^2 \theta} + \frac{d\phi_1^2}{72u^2 \cos^2 \theta} + \frac{d\phi_2^2}{72u^2 \sin^2 \theta} + \frac{1}{9} d\psi^2 \\ + \frac{i}{9u} d\psi \left( \frac{d\phi_1}{\cos \theta} + \frac{d\phi_2}{\sin \theta} \right) - \frac{1}{18u^2 \sin \theta \cos \theta} d\phi_1 d\phi_2, \quad (2.15)$$

which has a  $so(2)^3$  symmetry group and is degenerate as  $\theta \rightarrow 0, \pi/2$ .

All of the examples so far are symmetric negative curvature manifolds with degenerate boundaries. One could also start with a symmetric manifold which is degenerate as some parameter  $u$  is taken to zero. One would then re-interpret  $u$  as an effective radius and radially extend the hypersurface to construct an Einstein manifold in one higher dimension. This point of view was discussed in [22] in the context of negative curvature spacetimes with non-degenerate boundaries. It will become clearer in the following section how one would radially extend the hypersurface in the degenerate case.

The conformal symmetry group will be larger than in most of the examples given above, and so one will be able to fix more quantities in the conformal field theory. However, calculating bulk quantities will be correspondingly more difficult, since the bulk symmetry group will in general be smaller. This will be particularly relevant when trying to derive, for example, scalar correlation functions from bulk actions for massive scalar fields.

### III. REGULARISATION OF THE EUCLIDEAN ACTION

One of the most interesting developments arising from the AdS/CFT correspondence has been the use of counterterms in the Euclidean action to define the action independently of background for negative (and in certain limits zero) curvature manifolds [23], [11], [13]. Consider an Einstein manifold which satisfies the field equations derived from (2.2), whose metric near the boundary can be written in the form

$$ds^2 = \frac{dx^2}{l^2 x^2} + \frac{1}{x^2} \gamma_{ij} dx^i dx^j, \quad (3.1)$$

where  $\gamma$  is finite and non-degenerate on the boundary itself. In this section we will consider only four-dimensional spacetimes, since this is the dimensionality of the explicit examples which we will use. Following a theorem by Fefferman and Graham [24], [25] the conformal metric  $\gamma_{ij}$  can be written [23], [15] as

$$\gamma_{ij} = \gamma_{ij}^0 + x^2 \gamma_{ij}^2 + x^4 \gamma_{ij}^4 + \dots, \quad (3.2)$$

where  $\gamma^2$  is defined in terms of the curvature of  $\gamma^0$  as [22]

$$\gamma_{ij}^2 = -\frac{1}{4l^2} (R_{ij}^0 - \frac{1}{4} R^0 \gamma_{ij}^0), \quad (3.3)$$

and  $\gamma^4$  depends on fourth derivatives of  $\gamma^0$ .  $\gamma^0$  is independent of  $x$  when the conformal boundary is non-degenerate. Note that the choice of bulk conformal frame - or in other words, the magnitude of the cosmological constant - combined with the existence of a non-degenerate conformal boundary effectively fixes the coefficient of the  $dx^2$  term in (3.1) [24] to be  $1/l^2$ . We emphasise this point since it will be important in what follows.

We can then formally expand the Einstein action as [23]

$$\begin{aligned} I_{\text{bulk}} &= -\frac{1}{16\pi G_4} \int_{N_\epsilon} d^4x \sqrt{g} (R(g) + 6l^2) - \frac{1}{8\pi G_4} \int_{N_\epsilon} d^3x \sqrt{\gamma} K; \\ &= \frac{1}{16\pi G_4} \int_{N_\epsilon} d^3x \sqrt{\gamma^0} \left( -\frac{4l}{\epsilon^3} + \frac{16l}{\epsilon} \text{tr}(\gamma^0)^{-1} \gamma^2 + \dots \right); \\ &= -\frac{l}{4\pi G_4} \int_{N_\epsilon} d^3x \sqrt{\tilde{\gamma}} \left( 1 + \frac{1}{4l} R(\tilde{\gamma}) + \dots \right), \end{aligned} \quad (3.4)$$

where the inverse radius of the hypersurface over which we integrate,  $\epsilon \ll 1$ , is an IR regularisation parameter and  $x^2 \tilde{\gamma} = \gamma$ . The ellipses indicate non-divergent terms which we have omitted. The second equality is obtained by using the expansion of the boundary metric (3.2), (3.1) and integrating the bulk action explicitly. The key point is that since there are only a finite number of divergent terms one can introduce a local counterterm action  $I_{\text{ct}}$  dependent only on  $\tilde{\gamma}_{ij}$  and its covariant derivatives [11]

$$I_{\text{ct}} = \frac{l}{4\pi G_4} \int_{N_\epsilon} d^3x \sqrt{\tilde{\gamma}} (1 + \frac{1}{4l} R(\tilde{\gamma})). \quad (3.5)$$

Then the content of the AdS/CFT conjecture is that we make the identification that

$$I_{\text{bulk}} = W_{\text{cft}} + I_{\text{ct}}, \quad (3.6)$$

where we take the boundary to be the true conformal boundary and  $W_{\text{cft}}$  is the (finite) partition function for the conformal field theory. The purpose of this section is to show how and why this analysis breaks down when the boundary becomes degenerate.

### A. Degenerate boundaries and counterterm regularisation

For the manifolds with degenerate boundaries considered here, the expansion of  $\gamma_{ij}$  given in (3.2) breaks down; its derivation in fact relies on the existence of a non-degenerate conformal boundary of codimension one [24], [25]. We are going to consider a more general form for the expansion of the metric near the conformal boundary

$$ds^2 = \frac{dx^2}{l^2 x^2} + x^{-\delta} \gamma_{ij} dx^i dx^j. \quad (3.7)$$

We will assume that there is a well-defined expansion for the boundary metric of the form

$$\gamma_{ij} = \gamma_{ij}^0 + \gamma_{ij}^2 + \dots, \quad (3.8)$$

but  $\gamma^0$  will now depend on  $x$ . There is a preferred frame in which its determinant is independent of  $x$ : this choice of  $\gamma^0$  is natural if one requires that both the determinant and the inverse metric are well-defined as  $x \rightarrow 0$ . We assume from here on that this choice of normalisation for  $\gamma^0$  is imposed in the metric (3.7), which along with the normalisation of the coefficient of  $dx^2$  in (3.8), effectively determines the choice of the coordinate  $x$  given an Einstein metric satisfying (2.2).  $\gamma^2$  will be subleading in the sense that its determinant behaves as a positive power of  $x$  as one takes the limit  $x \rightarrow 0$ . As we will see later on in this section, the explicit metrics given in §II all admit an expansion of this form.

Suppose that  $\gamma^0$  effectively degenerates to a  $p$ -dimensional metric in the limit that  $x \rightarrow 0$ , where  $p$  will be determined by the number of independent vielbeins. Note that the  $p$ -dimensional metric does not in general have a non-zero determinant nor will the associated vielbeins be closed. In fact, the Bergman metric degenerates to a metric with zero determinant whose associated single vielbein is not closed. Then a typical degenerate boundary metric  $\gamma^0$  might be written as

$$\gamma_{ij}^0 = x^{-1} h_{ij}^{(p)} + x^{\frac{p}{3-p}} h_{ij}^{(3-p)}, \quad (3.9)$$

where  $h^q$  is of dimension  $q$  in the sense defined above. Now  $\gamma^2$  is defined by requiring that the metric (3.7) satisfies the Einstein equations expanded out in powers of  $x$ . By analysing the field equations, however, we find that although  $\gamma^2$  is well-defined, it is not a covariant tensor: it cannot be written in terms of  $\gamma^0$  and its curvature invariants. The definition of  $\gamma^2$  for (3.9) is not particularly illuminating since it is not generic; it involves second derivatives of  $h^p$  and  $h^{3-p}$  as one would expect.

Using the asymptotic form for the metric, the Einstein action for a manifold with a degenerate boundary (3.7) can then be written as

$$\begin{aligned} I_{\text{bulk}} &= -\frac{1}{16\pi G_4} \int_{N_\epsilon} d^4x \sqrt{g} (R(g) + 6l^2) - \frac{1}{8\pi G_4} \int_{N_\epsilon} d^3x \sqrt{\gamma} K, \\ &= \frac{l}{16\pi G_4} \int_{N_\epsilon} d^3x \sqrt{\tilde{\gamma}} \left( \frac{4}{\delta} - 3\delta \right) + I_{\text{nl}}, \end{aligned} \quad (3.10)$$

where the second equality follows from explicitly substituting the metric (3.7) into the action and integrating.  $\tilde{\gamma} = \epsilon^{-\delta} \gamma$  is the metric induced on a codimension one hypersurface of constant  $\epsilon$  and the integral is taken over a hypersurface of constant  $\epsilon \ll 1$ .

The second part of the action,  $I_{\text{nl}}$ , includes non-local terms and cannot be expressed covariantly in terms of the boundary metric. This term in the action is not in general finite, but diverges as one takes the limit  $\epsilon \rightarrow 0$ . However, the leading order divergence of the bulk action (which behaves as  $\epsilon^{-3\delta/2}$ ) can be removed by subtracting the first term in (3.10); this follows from the condition that  $\gamma^2$  is subleading to  $\gamma^0$ . One will be left with a leading order divergent term in  $I_{\text{nl}}$  which behaves as  $\epsilon^{-a}$  with  $a < 3\delta/2$ . Note that the first term in the action (3.10) agrees with that for non-degenerate boundaries when one takes  $\delta = 2$ .

One implication of the above is that one cannot introduce local counterterms to remove the divergence of the bulk action as  $\epsilon \rightarrow 0$ . Suppose we tried to take a counterterm action of the form [11]

$$I_{\text{ct}} = \frac{1}{16\pi G_4} \int_{N_\epsilon} d^3x \sqrt{\tilde{\gamma}} [a_0[\delta] + a_1 R(\tilde{\gamma}) + a_2 (R(\tilde{\gamma}))^2 + b_2 R^{ij}(\tilde{\gamma}) R_{ij}(\tilde{\gamma}) + \dots]. \quad (3.11)$$

Provided we pick the first coefficient  $a_0[\delta]$  according to (3.10) we can remove the leading order divergence - but there is no generic way to define the other coefficients. In fact, even choosing  $a_0[\delta]$  in this way really represents a fine-tuning which we are not allowed to do. One more general grounds, we can see that this series cannot be convergent in  $\epsilon$  without adjusting the coefficients to each solution. Since the curvature invariants of hypersurfaces of constant  $\epsilon$  are of order  $l$  for degenerate boundaries (compared to invariants of order  $\epsilon$  to positive powers for non-degenerate boundaries), there is no small expansion parameter and no reason for the series to converge.

This behaviour of the curvature invariants follows from the Gauss-Codacci condition for the induced hypersurface

$$R(\tilde{\gamma}) = (K^2 - K_{ab} K^{ab} - 6l^2), \quad (3.12)$$

where  $K_{ab}$  is the extrinsic curvature of the hypersurface and  $K$  is its trace as before. For a metric which can be written in the form (3.2) with  $\gamma^0$  non-degenerate, then



$$K^2 = 9l^2 + \mathcal{O}(\epsilon^2); \quad K_{ab}K^{ab} = 3l^2 + \mathcal{O}(\epsilon^2), \quad (3.13)$$

and so the curvature invariants of the hypersurface behave as positive powers of  $\epsilon$ , which is really the basis of the counterterm subtraction procedure [11]. However, if  $\gamma^0$  is degenerate and, for example, of effective dimension  $p$ , then

$$K^2 = \frac{(3+p)^2}{4}l^2 + \mathcal{O}(\epsilon^2); \quad K_{ab}K^{ab} = \frac{3(p+1)}{4}l^2 + \mathcal{O}(\epsilon^2), \quad (3.14)$$

and so, as previously mentioned, the curvature of the hypersurface is *negative* and of order  $l$ . It is a generic feature of spaces with degenerate conformal boundary that the induced metric on the boundary has finite negative curvature, rather than an infinite curvature radius as is usual. We should perhaps mention here that some of the analysis of the AdS/CFT correspondence relies on non-negative curvature of the CFT background spacetime. In particular, the discussion in [26] relies specifically on a conformal boundary of positive scalar curvature. Interesting issues that arise even in the non-degenerate case when the boundary has negative curvature are discussed in [27].

Of course after a little reflection we should not be surprised that local counterterms cannot remove the divergence of the bulk action. Since  $\epsilon$  appears explicitly in the conformal field theory background geometry, we cannot expect the partition function to be independent of this parameter. Furthermore we should probably expect the partition function to diverge as the geometry becomes degenerate.

As a simple example let us consider a generic conformal field theory on a  $d$ -dimensional background

$$ds^2 = u^{d-1}d\tau^2 + u^{-1}h_{ij}dx^i dx^j, \quad (3.15)$$

where  $\tau$  is the trivially fibered imaginary time coordinate with period  $\beta$  and  $h$  is a non-degenerate metric. We suppose that  $u$  corresponds to a radial parameter in the bulk theory, and this metric is conformal to that induced on hypersurfaces of constant  $u$ . As  $u \rightarrow 0$ , the metric will become degenerate, although in this (preferred) conformal frame the determinant remains regular. Suppose we now conformally rescale the metric such that

$$\tilde{ds}^2 = d\tilde{\tau}^2 + h_{ij}dx^i dx^j, \quad (3.16)$$

where we have defined a new imaginary time coordinate  $\tilde{\tau} = u^{\frac{d}{2}}\tau$ . In this conformal frame it is trivial to write down the main dependence of the partition function since the effective temperature is high in the degenerate limit:  $\tilde{\beta} \rightarrow 0$  as  $u \rightarrow 0$ . This means that the partition function for the conformal field theory behaves as

$$W_{\text{cft}} \sim T^{d-1}u^{-\frac{d(d-1)}{2}}, \quad (3.17)$$

where  $T$  is the inverse of  $\beta$ , and would be interpreted as the finite temperature of the bulk theory. Thus the partition function does indeed diverge as the geometry becomes degenerate.

Given a generic  $d$ -dimensional metric which becomes degenerate in the sense considered here as some parameter  $u \rightarrow 0$  we can construct a  $(d+1)$ -dimensional metric satisfying the equations derived from (2.2) as follows. Firstly, we should find the conformal frame in which the  $d$ -dimensional metric determinant is independent of  $u$ . Then we should write the higher-dimensional metric in the form (3.7) and fix  $\delta$  from the leading order terms in the Einstein equations.  $\gamma^2$  will follow from an expansion in powers of  $u$ .

## B. Interpretation of the bulk action

If one cannot remove all the divergences in the action with covariantly defined counterterms, one has to decide how to interpret the bulk action in terms of the dual conformal field theory. One suggestion - close in spirit to interpretations of the Randall-Sundrum scenario in terms of the AdS/CFT correspondence [14], [15], [17] - is the following. Instead of the ultimate goal being to take the  $\epsilon \rightarrow 0$  limit so that the cutoff boundary becomes the true boundary, we need to keep the boundary at finite  $\epsilon$ . This will ensure that the background geometry for the dual conformal field theory is non-degenerate.

In the Randall-Sundrum scenario [14] an Einstein term is induced into the effective action on the hypersurface, plus a cosmological term which we can effectively adjust to zero by adding a brane tension term [15]. The presence of these two terms in the induced action is manifest from the counterterm action (3.5). However, for degenerate boundaries there is no Einstein term in the “hypersurface” action. The leading order propagator will follow from differentiating the action twice with respect to the hypersurface metric  $\tilde{\gamma}$ . Unsurprisingly one can’t get a sensible brane world scenario from a higher-dimensional metric with degenerate boundary.

The natural suggestion for the correspondence between the bulk and conformal field theory partition functions is that we should simply take

$$I_{\text{bulk}}(\epsilon) \approx W_{\text{cft}}(\epsilon), \quad (3.18)$$

where  $W_{\text{cft}}(\epsilon)$  is the partition function for the conformal field theory in a geometry regulated by  $\epsilon$ . Following the discussion in the last subsection, one might question why we don’t take the correspondence to be instead

$$I_{\text{nl}}(\epsilon) \sim W_{\text{cft}}(\epsilon), \quad (3.19)$$

where we have removed the leading order divergence of the bulk action by subtracting a counterterm of the form (3.10). However subtraction of such a counterterm would not be satisfactory from a holographic point of view, since one would need to know the index  $\delta$  to carry out the subtraction *but*  $\delta$  is not known by the conformal boundary geometry. Another way of saying this is that one effectively has to adjust the subtraction to the bulk geometry rather than taking a generic subtraction. Of course in the non-degenerate case one still needs to know that  $\delta = 2$  to carry out the subtraction but the regularity of the geometry of the regularisation limit implicitly tells us that  $\delta = 2$ .

This proposal for the correspondence is equivalent to taking a strong version of the holographic principle [5]: it assumes that quantum gravity on any volume contained within a manifold can be described by a theory defined on the boundary of the volume. This is the basis for the recent work of [15], [12], [16] but our proposal extends this principle to more general negative curvature manifolds. One should be able to make more precise the correspondence between the bulk field equations and the renormalisation group flow equations in the conformal field theory along the lines of [12]. The difference will be that, in addition to the renormalisation group flow in the conformal field theory as one flows in from infinity, one will also have a flow in the effective target space geometry for the conformal field theory.

We should also mention that, although this procedure for cutting off the interior path integral at a finite boundary seems to be the right thing to do to compare partition functions, we probably need to be more careful about how we do this. Simply cutting off the theory will throw out some physics since it will not tell us about physical processes in which particles propagate across our cutoff boundary. However, our naive approach will be adequate for the discussions here.

To derive other thermodynamic quantities in the boundary conformal field theory from the bulk, one would need to use the quasilocal tensor defined by Brown and York as [33]

$$T^{\mu\nu} = \frac{2}{\sqrt{\tilde{\gamma}}} \frac{\delta I_{\text{bulk}}}{\delta \tilde{\gamma}_{\mu\nu}}, \quad (3.20)$$

and then define conserved quantities associated with Killing vectors  $\xi$  as

$$Q_\xi(\epsilon) = \int_{\Sigma_\epsilon} d^2x \sqrt{\sigma} T_{\mu\nu} u^\mu \xi^\nu, \quad (3.21)$$

where  $u$  is the unit normal to a hypersurface  $\Sigma_\epsilon$  in  $N_\epsilon$ . The thermodynamic relation between these quantities would be defined as usual as

$$I_{\text{bulk}}(\epsilon) = \beta M(\epsilon) + \dots - S(\epsilon), \quad (3.22)$$

where  $\beta$  is the inverse temperature, and  $M(\epsilon)$  and  $S(\epsilon)$  correspond to the mass and entropy respectively of the regulated conformal field theory.

### C. Example 1: The Bergman metric

To check whether the bulk/boundary partition functions do diverge in the same way, let us try to calculate both for some of the metrics discussed in §II. Suppose we introduce into the Bergman metric an IR cutoff  $\sinh \rho = lR \gg 1$  so that the boundary geometry is conformal to

$$ds^2 = l^2 R^2 (d\psi + \cos \theta d\phi)^2 + (d\theta^2 + \sin^2 \theta d\phi^2); \quad (3.23)$$

Then the bulk Euclidean action is

$$I = -\frac{5\pi l^2 R^4}{4G_4} - \frac{3\pi R^2}{2G_4}. \quad (3.24)$$

To calculate the surface term in (3.10), we need to bring the metric near the conformal boundary into the form (3.7). Defining  $x = 2e^{-\sqrt{2}\rho}$  then the leading order terms in the metric are

$$ds^2 = \frac{dx^2}{l^2 x^2} + \frac{x^{-\frac{4\sqrt{2}}{3}}}{2l^2} \{x^{-\frac{2\sqrt{2}}{3}} (d\psi + \cos \theta d\phi)^2 + x^{\frac{\sqrt{2}}{3}} (d\theta^2 + \sin^2 \theta d\phi^2)\}, \quad (3.25)$$

from which we see that we must take  $\delta = 4\sqrt{2}/3$  in (3.7), and hence the first term in (3.10) becomes

$$I_{\text{surf}} = -\frac{5\pi l^2 R^4}{4G_4} + \dots, \quad (3.26)$$

which as expected coincides with the leading order divergence of the effective action.

As usual, strong coupling prevents us from calculating the partition function for the associated conformal theory on the squashed three sphere directly; however, in this case, we can calculate the  $R$  dependence by an indirect method. Supergravity in negative curvature Taub-Nut and Taub-Bolt manifolds [28] also corresponds to the  $(2+1)$  dimensional “exotic” conformal field theory [29] which lives on the world volume of M2-branes after placing them on a squashed three sphere. There is of course a very close relationship between the AdS Taub-Bolt manifolds and the Bergman metric. The Bergman metric is a radial extension of the second power of the Hopf bundle over  $S^2$  whilst the nut and bolt metrics are radial extensions of the first power of the Hopf bundle over  $S^2$  [18]. There is no problem in calculating the regularised Euclidean action for the Taub-Nut and Taub-Bolt manifolds which have non-degenerate boundaries. The metric for the nut solution <sup>1</sup> is

$$ds^2 = V(r)(d\tau + 2n \cos \theta d\phi)^2 + V^{-1}(r)dr^2 + (r^2 - n^2)(d\theta^2 + \sin^2 \theta d\phi^2);$$

$$V(r) = \frac{(r-n)(l^2 r^2 + 2nl^2 r + 1 - 3n^2 l^2)}{(r+n)}, \quad (3.27)$$

and the action was calculated using counterterm subtraction in [30]

$$I = \frac{4\pi n^2}{G_4}(1 - 2n^2 l^2), \quad (3.28)$$

with the boundary geometry behaving as

$$ds^2 = 4n^2 l^2 (d\psi + \cos \theta d\phi)^2 + (d\theta^2 + \sin^2 \theta d\phi^2), \quad (3.29)$$

where we identify  $\tau \equiv \psi n$ . The usual dictionary for the AdS/CFT correspondence [1] implies that we should take

$$N^{\frac{3}{2}} \approx \frac{1}{l^2 G_4}, \quad (3.30)$$

where  $N$  is a measure of the number of unconfined degrees of freedom for the gauge theory describing the dynamics of  $N$  parallel M2-branes wrapped on a squashed three sphere. So to compare the conformal field theory in the background geometry (3.29) with that in (3.23) we need to take the same values of  $l^2 G_4$  and set  $R = 2n$ . In this limit the conformal field theory partition function behaves as

$$I = \frac{\pi R^2}{G_4} \left(1 - \frac{R^2 l^2}{2}\right). \quad (3.31)$$

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<sup>1</sup>The bolt solution does not exist in the parameter range relevant here.

In the extreme squashing limit, the action diverges in the same way as the bulk action for the Bergman metric. Of course, we shouldn't expect the coefficients to agree, since we can't assume that the two spacetimes correspond to the same state in the conformal field theory <sup>2</sup>. However, since the degeneracy of the geometry will determine the leading order divergence of the partition function, we should expect the actions to diverge in the same way as we take  $R$  to infinity.

There is a possible flaw in the above argument. It is not obvious that we can regulate the action for the nut spacetime and then take a singular limit in  $n$ ; these operations do not necessarily commute, since the spacetime becomes very singular as  $n \rightarrow \infty$ . Although we should be reassured that a very similar limiting process appears to work when one calculates the action for critically rotating black holes [31], it would nice to check the above conclusions in another way.

Since the leading order behaviour of the partition function should not depend on the details of the conformal field theory as  $R \rightarrow \infty$  it should be reproduced by the partition function for free conformally coupled scalar and spinor fields in this background. A related calculation was carried out in [32]; it was found that if one considered eigenmodes of a scalar field on a squashed sphere satisfying

$$(-\nabla + \frac{1}{4})\Phi_k = \lambda_k \Phi_k, \quad (3.32)$$

then the partition function obtained from the zeta function  $\zeta(s) = \sum_k \lambda_k^{-s}$  did indeed behave as  $R^4$  in the extreme oblate limit. However, this calculation is not directly relevant to conformally coupled fields <sup>3</sup>: in the large  $R$  limit, the operator (3.32) is very different from the conformally invariant operator

$$(-\nabla + \frac{R_g}{8}) = (-\nabla + \frac{1}{4} - \frac{R^2}{16}), \quad (3.33)$$

where  $R_g$  is the Ricci scalar. It is not difficult to apply the same techniques to show that the divergence as  $R^4$  persists for the conformally coupled operator (3.32); the analysis mirrors that of [32] and is summarised in the Appendix. Note that there doesn't seem to be any natural intuitive explanation for the  $R^4$  dependence of the partition function; it follows in a non-trivial way from the geometry.

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<sup>2</sup>Indeed, if we accept the hypothesis (3.18) as true, then the entropy for the Bergman metric is positive whereas that for the nut solutions is negative, so the Bergman metric corresponds to a highly excited state. Of course the use of this argument is circular. Note that the negativity of the entropy for the nut solution can be viewed as a manifestation of the pathologies in the causal structure as discussed in §2.

<sup>3</sup>as Andy Strominger has also pointed out to me.

### D. Example 2: $H^2 \times H^2$

The second example we will consider is the product of two hyperbolic spaces  $H^2 \times H^2$  whose Einstein action is

$$I = -\frac{1}{48\pi G_4 l^2} \int \frac{d\theta dx_1 dx_2}{u^2 \sin^2 \theta \cos^2 \theta} = -\frac{\sqrt{2}\sigma_3}{48\pi G_4 l^2 u^2}, \quad (3.34)$$

where we have introduced a regulated volume  $\sigma_3$  for the volume of non-compact hypersurfaces of constant  $u$  in the induced boundary metric

$$ds^2 = \frac{u^{\frac{4}{3}} d\theta^2}{2 \sin^2 \theta \cos^2 \theta} + \frac{dx_1^2}{u^{\frac{2}{3}} \cos^2 \theta} + \frac{dx_2^2}{u^{\frac{4}{3}} \sin^2 \theta}. \quad (3.35)$$

The metric can be brought into the form (3.7) with the choice of coordinate

$$x = u^{\frac{\sqrt{2}}{\sqrt{3}}} \sin^{\frac{1}{\sqrt{6}}} \theta \cos^{\frac{1}{\sqrt{6}}} \theta, \quad (3.36)$$

and hence the above analysis is applicable here. Since there is in this case no obvious supergravity background with a related conformal boundary for which we can also calculate the action, the best that we can do is to check whether we can reproduce this form of the partition function from conformally coupled scalars in the background (3.35). The Ricci scalar for this metric is

$$R_g = -4 - 4 \cos^2 \theta \sin^2 \theta, \quad (3.37)$$

and modes of a conformally coupled scalar field behaving as  $\phi(\theta, x_1, x_2) \sim \phi_\alpha(\theta) e^{ik_i x_i}$  satisfy the equation

$$\begin{aligned} (2 \cos^2 \theta \sin^2 \theta \partial_\theta^2 - \cos^2 \theta k_1^2 u^2 - \sin^2 \theta k_2^2 u^2 + \frac{1}{2} + \frac{1}{2} \cos^2 \theta \sin^2 \theta) \phi_\alpha(\theta) \\ = -\lambda(\alpha, k_i u) \phi_\alpha(\theta). \end{aligned} \quad (3.38)$$

In fact we don't need to find the eigenvalues explicitly; all we need to know is that the eigenvalues  $\lambda$  depend only on the combinations  $(k_i u)$  and  $\alpha$ . Furthermore, since the domain over which we are solving the equation is non-compact, the index  $\alpha$  is continuous and the zeta function summation will take the form

$$\begin{aligned} \zeta(s) &= \sum \lambda^{-s} = \sigma_3 \int d\alpha dk_i \lambda(\alpha, k_i u)^{-s}; \\ &= \sigma_3 u^{-2} \tilde{\zeta}(s), \end{aligned} \quad (3.39)$$

where  $\sigma_3$  is again the regulated volume and in the latter equality  $\tilde{\zeta}(s)$  is a function only of  $s$ . Since the partition function can depend only on  $\zeta'(0)$ , it manifestly exhibits the same behaviour as the bulk action (3.34), in agreement with our suggestion for the interpretation of the bulk action. Suppose we interpret  $x_1$  as the Euclidean time direction; then the thermodynamic relation is given by

$$I_{\text{bulk}}(u) = \beta_{x_1} M(u), \quad (3.40)$$

where  $\beta_{x_1}$  is the inverse temperature and the cutoff mass  $M(u)$  is negative. The entropy vanishes, which implies that  $H^2 \times H^2$  corresponds in some sense to the ground state of the conformal field theory, but the energy is negative which we should probably interpret as discussed in [27].

## IV. CORRELATION FUNCTIONS IN THE BOUNDARY CFT

In the previous section we considered how the bulk action corresponds to the partition function for the conformal field theory. The next question to ask is how the bulk supergravity action acts as a generating functional for the correlation functions of the conformal field theory. The analysis for the Bergman metric was carried out in [10]; the  $su(2, 1)$  symmetry of the bulk corresponds to a  $su(2, 1)$  conformal symmetry group of the boundary. This conformal symmetry is enough to fix the functional form of two-point functions of scalar operators entirely, and this form is reproduced from the action for bulk scalar fields. In particular, as in other cases of the AdS/CFT correspondence, fields of a particular mass  $m$  and spin  $s$  are found to correspond to scalar operators of definite conformal weights  $\Delta(m, s)$  in the boundary CFT.

Manifolds of degenerate boundary which fall into the first category of §II can hence be dealt with in much the same way as in the usual AdS/CFT correspondence. However, the analysis is different for spaces falling into the second category. These spaces are characterised by the existence of more than one infinite direction, not linked by the symmetry group. As we will discuss in this section, this means that massive scalar fields will give rise to boundary data which is a sum of data of different conformal weights; the relationship between the mass and the conformal weight in the CFT is more subtle. A secondary characteristic of these spaces is that the conformal symmetry group is not large enough to fix the form of even the two point functions completely.

### A. Two point functions from conformal symmetry

We will consider here the simplest non-trivial example,  $H^2 \times H^2$ . Since the symmetry group of the manifold is  $sl(2, R) \times sl(2, R)$ , which has a maximal compact subgroup of  $so(2) \times so(2)$ , the boundary has only the latter group of symmetries. Expressed in terms of the  $(u, \theta)$  coordinates, the Killing vectors in the bulk are

$$\begin{aligned} k_1 &= \partial_{x_1}; & k_2 &= \partial_{x_2}; \\ l_1 &= x_1 \partial_{x_1} + u \cos^2 \theta \partial_u - \cos \theta \sin \theta \partial_\theta; \\ l_2 &= x_2 \partial_{x_2} + u \sin^2 \theta \partial_u + \cos \theta \sin \theta \partial_\theta; \\ m_1 &= (x_1^2 - u^2 \cos^2 \theta) \partial_{x_1} + 2x_1 u \cos^2 \theta \partial_u - 2x_1 \cos \theta \sin \theta \partial_\theta; \\ m_2 &= (x_2^2 - u^2 \sin^2 \theta) \partial_{x_2} + 2x_2 u \sin^2 \theta \partial_u + 2x_2 \cos \theta \sin \theta \partial_\theta. \end{aligned} \tag{4.1}$$

If one restricts to the boundary  $u \rightarrow 0$ , then the  $k_i$  remain symmetries but the  $l_i$  are conformal symmetries only. Notice that one does not need the inverse metric to define the conformal Killing vector equations and hence the conformal symmetries are well defined even without a non-degenerate metric.

Now let us consider how the two-point function of scalar fields  $\langle \mathcal{O}_{\Delta_1}(x) \mathcal{O}_{\Delta_2}(\bar{x}) \rangle$  is fixed by the requirement of invariance under conformal transformations. Under a conformal transformation generated by  $\xi$  a field of conformal weight  $\Delta$  will transform as

$$\delta_\xi \mathcal{O} = (\mathcal{L}_\xi + \frac{\Delta}{3} D_m \xi^m) \mathcal{O}, \tag{4.2}$$

where  $\mathcal{L}$  is the Lie derivative. Then the requirement of invariance under the isometries  $k_i$  implies that the two-point function only depends on the translationally invariant quantities  $(x_1 - \bar{x}_1)$  and  $(x_2 - \bar{x}_2)$ . The requirement for the two-point function to be covariant under the transformations generated by the  $l_i$  and  $m_i$  is

$$[l_i^{(x)} + l_i^{(y)}] \langle \mathcal{O}_{\Delta_1}(x) \mathcal{O}_{\Delta_2}(\bar{x}) \rangle = -\frac{1}{3} [\Delta_1 D_m l_i^{m(x)} + \Delta_2 D_m l_i^{m(y)}] \langle \mathcal{O}_{\Delta_1}(x) \mathcal{O}_{\Delta_2}(\bar{x}) \rangle. \quad (4.3)$$

Now in this equation we need the inverse metric to be finite in order to define the right-hand side. To do this we note that if we use the conformally rescaled boundary metric (3.35) discussed in §III then the metric determinant is independent of  $u$  and

$$D_m l_i^{m(x)} = \sin^2 \theta \cos^2 \theta \partial_m (\sin^{-2} \theta \cos^{-2} \theta l_i^m). \quad (4.4)$$

Note that both the non-degenerate measure and the metric itself have conformal dimension of minus three. Using the four conformal covariance conditions we can constrain the two-point function to be of the form

$$\langle \mathcal{O}_{\Delta_1}(x) \mathcal{O}_{\Delta_2}(\bar{x}) \rangle = \int d\chi f(\chi, \Delta_1, \Delta_2) \frac{\sin^{\frac{2\Delta_1}{3}-\frac{\chi}{2}} \theta \sin^{\frac{2\Delta_2}{3}-\frac{\chi}{2}} \bar{\theta} \cos^{\frac{\chi}{2}} \theta \cos^{\frac{\chi}{2}} \bar{\theta}}{(x_1 - \bar{x}_1)^\chi (x_2 - \bar{x}_2)^{\frac{2\Delta_1}{3} + \frac{2\Delta_2}{3} - \chi}}. \quad (4.5)$$

As expected the conformal symmetry group is not large enough to fix the form of the two-point function completely. The function  $f$  is not fixed by symmetry and furthermore conformal invariance does not fix  $\Delta_1 = \Delta_2$ ; fields of unequal conformal weight are not excluded from having a non-zero correlation function.

## B. Scalar fields in the bulk

Now let us consider how this form for the two-point function is reproduced by the bulk theory. One of the most interesting differences between this bulk boundary correspondence and the usual non-degenerate correspondence is that a bulk scalar field of mass  $m$  does not correspond to a single operator of weight  $\Delta(m)$ . Instead, the scalar field acts as a source for a set of operators of weights which depend not only on  $m$  but also on the “mode” of the scalar field.

One can easily understand how this arises by looking at explicit solutions of the field equation. The field equation for a free scalar field of mass  $m$  is

$$[z_1^2(\partial_{z_1}^2 + \partial_{x_1}^2) + z_2^2(\partial_{z_2}^2 + \partial_{x_2}^2) - m^2] \Phi^m = 0, \quad (4.6)$$

and so modes of the field behave as

$$\Phi^m \sim (z_1 z_2)^{1/2} K_\nu(k_1 z_1) K_{\sqrt{m^2 - \nu^2}}(k_2 z_2) e^{ik_1 x_1 + ik_2 x_2}, \quad (4.7)$$

where we have chosen Bessel functions such that modes are bounded at the interior points  $z_1, z_2 \rightarrow \infty$ . The allowed values of  $m$  are determined by considering the spectrum of supergravity on  $S^7$ ; we find that (in the units used here)  $m^2 \geq -3/8$ . We then restrict the allowed values of  $\nu$  to  $\text{Re}(\nu) > 0$  to enforce boundedness in the interior. Note that most



modes will consist of decaying oscillations in one hyperbolic space and exponential decay in the other.

We should briefly mention that since the space is supersymmetric there are no unstable fluctuations of the scalar field; the point is that although modes may be normalisable on one space they cannot be simultaneously normalisable on the other. If one considers eigenmodes of  $\Phi(m)$  with eigenvalues  $\lambda_k$  then modes of negative  $\lambda_k$  are not normalisable.

A more elegant way of expressing the above analysis is in terms of representation theory. Solutions of the wave equation for a massive scalar field form a representation of  $sl(2, R) \times sl(2, R)$ , which can be decomposed as products of representations of  $sl(2, R)$  and  $sl(2, R)'$  with Casimirs proportional to  $\nu^2$  and  $(m^2 - \nu^2)$  respectively. Suppose we then consider primary fields satisfying  $k_1 \Psi = k_2 \Psi = 0$ ,  $l_1 \Psi = -h_1 \Psi$  and  $l_2 \Psi = -h_2 \Psi$ , which behave as

$$\Psi \sim u^{-h_1-h_2} \cos^{-h_1} \theta \sin^{-h_2} \theta. \quad (4.8)$$

The quadratic Casimir is

$$m^2 \Psi = (l_1^2 + l_2^2 - \{k_1, m_1\} - \{k_2, m_2\}) \Psi = [h_1(h_1 + 1) + h_2(h_2 + 1)] \Psi. \quad (4.9)$$

The conformal weights with respect to the two  $sl(2, R)$  conformal groups are thus related by the mass of the bulk scalar field, but are not fixed; this is the origin of the  $\chi$  integration in (4.5). For fields of arbitrary spin  $s$  the mass and conformal weight relation (4.9) becomes

$$m^2 = [h_1(h_1 + 1) + h_2(h_2 + 1)] - \frac{s^2}{2}. \quad (4.10)$$

Rewriting the scalar field in terms of the  $(u, \theta)$  variables and taking the limit  $u \rightarrow 0$  we get

$$\Phi^m \rightarrow u^{1-\nu-\sqrt{m^2-\nu^2}} (\cos^{1/2-\nu} \theta \sin^{1/2-\sqrt{m^2-\nu^2}} \theta k_1^{-\nu} k_2^{-\sqrt{m^2-\nu^2}} e^{ik_1 x_1 + ik_2 x_2}). \quad (4.11)$$

Note that the  $u$  dependence is in general complex depending on the value of  $\nu$ . Explicitly the conformal weights  $h_i$  are given in these variables by

$$h_1 = \nu - \frac{1}{2}; \quad h_2 = \sqrt{m^2 - \nu^2} - \frac{1}{2}. \quad (4.12)$$

The total conformal weight of the boundary data will be determined by the  $u$  dependence and is not independent of  $\nu$ ; hence different modes of a massive field will give rise to boundary data of different conformal weight.

The action for a free massive scalar field reduces to the boundary term:

$$I^{(m)} = \int dx dy \frac{d\theta}{u \sin^2 \theta \cos^2 \theta} \Phi^m \partial_u \Phi^m, \quad (4.13)$$

where in this equation and all that follow we are suppressing constant factors. Fourier transforming (4.7), the massive scalar field can be written in terms of propagators on each hyperbolic space as

$$\Phi^{(m)} = \int d\nu d\bar{x}_1 d\bar{x}_2 u^{\alpha+\beta} \frac{\cos^\alpha \theta}{(u^2 \cos^2 \theta + (\Delta x_1)^2)^\alpha} \frac{\sin^\beta \theta}{(u^2 \sin^2 \theta + (\Delta x_2)^2)^\beta} \Phi^{(m)}(\nu, \bar{x}_1, \bar{x}_2), \quad (4.14)$$

where  $\alpha = \frac{1}{2} + \nu$  and  $\beta = \frac{1}{2} + \sqrt{m^2 - \nu^2}$ . In the limit  $u \rightarrow 0$ ,

$$\Phi^{(m)} \rightarrow \int d\nu u^{1-\nu-\sqrt{m^2-\nu^2}} \cos^\alpha \theta \sin^\beta \theta \Phi^{(m)}(\nu, x_1, x_2). \quad (4.15)$$

In these expressions we are drawing on the by now well known propagators first discussed in [3]. Note that we have not corrected the normalisation of the propagators following [34] to ensure the right coefficients as  $u \rightarrow 0$ ; in this expression, and all that follow, we will ignore  $\nu$  dependent normalisation factors. We should allow the  $\nu$  integration to run over all possible values. Furthermore,

$$\begin{aligned} \partial_u \Phi^{(m)} \rightarrow \int d\nu d\bar{x}_1 d\bar{x}_2 & \left[ \frac{(\alpha + \beta) u^{\alpha+\beta-1}}{(\Delta x_1)^{2\alpha} (\Delta x_2)^{2\beta}} - \frac{2\alpha u^{1+\alpha+\beta} \cos^2 \theta}{(\Delta x_1)^{2(\alpha+1)} (\Delta x_2)^{2\beta}} \right. \\ & \left. - \frac{2\beta u^{1+\alpha+\beta} \sin^2 \theta}{(\Delta x_1)^{2\alpha} (\Delta x_2)^{2(\beta+1)}} \right] \cos^\alpha \theta \sin^\beta \theta \Phi^{(m)}(\nu, x_1, x_2) \end{aligned} \quad (4.16)$$

Since  $\text{Re}(\alpha) > 0$  and  $\text{Re}(\beta) > 0$ , the first term is of leading order as  $u \rightarrow 0$ . However, as we shall see below, we cannot neglect the subleading terms in this case, since these will give finite contributions to two point functions even as  $u \rightarrow 0$ .

It is convenient at this stage to rewrite the last two integrals as integrals over conformal weight of the boundary data, since we will eventually want to compare predictions for two point functions with the boundary theory expectations. Then

$$\Phi^{(m)}(u, \theta, x_1, x_2) \rightarrow \int d\lambda u^{-2\lambda/3} Y^{(m)}(\lambda, \theta) \Phi^{(m)}(\lambda, x_1, x_2), \quad (4.17)$$

where as we will see  $-\lambda$  is the conformal weight of the boundary data and we introduce the “eigenfunctions”

$$Y^{(m)}(\lambda, \theta) = \cos^\alpha \theta \sin^\beta \theta, \quad (4.18)$$

where  $(\alpha + \beta) = \frac{2\lambda}{3} + 2$  and in addition

$$\left(\alpha - \frac{1}{2}\right)^2 = m^2 - \left(\beta - \frac{1}{2}\right)^2. \quad (4.19)$$

It is also helpful to introduce the notation

$$K^{(m)}(\alpha, \beta, \theta, \Delta x_1, \Delta x_2) = \frac{\cos^\alpha \theta \sin^\beta \theta}{(\Delta x_1)^{2\alpha} (\Delta x_2)^{2\beta}}, \quad (4.20)$$

and to simplify notation further we will suppress coordinate dependence where obvious from now on. Then,

$$\begin{aligned} \partial_u \Phi^{(m)} \rightarrow \int d\lambda d\bar{x}_1 d\bar{x}_2 & \left[ \left(2 + \frac{2\lambda}{3}\right) u^{1+2\lambda/3} K^{(m)}(\alpha, \beta) - 2\alpha u^{3+\frac{4\lambda}{3}} K^{(m)}((\alpha+1), \beta) \cos \theta \right. \\ & \left. - 2\beta u^{3+\frac{4\lambda}{3}} K^{(m)}(\alpha, (\beta+1)) \sin \theta \right] \Phi^{(m)}(\lambda, \bar{x}_1, \bar{x}_2). \end{aligned} \quad (4.21)$$

This action is of the form

$$I^{(m)} = \int d\sigma d\lambda d\bar{x}_1 d\bar{x}_2 d\bar{\lambda} u^{2\bar{\lambda}/3-2\lambda/3} Y^{(m)}(\lambda, \theta) \Phi^{(m)}(\lambda) \left\{ \left(2 + \frac{2\bar{\lambda}}{3}\right) K^{(m)}(\bar{\alpha}, \bar{\beta}) - 2u^2 \left( \bar{\alpha} K^{(m)}(\bar{\alpha} + 1, \bar{\beta}) + \bar{\beta} K^{(m)}(\bar{\alpha}, \bar{\beta} + 1) \right) \right\} \Phi^{(m)}(\bar{\lambda}), \quad (4.22)$$

where  $d\sigma$  is the non-degenerate measure on the boundary and  $\alpha, \beta$  satisfy the constraints  $(\bar{\alpha} + \bar{\beta}) = 2 + \frac{2\bar{\lambda}}{3}$  as well as the constraint (4.19).

The bulk/boundary correspondence tells us that the bulk scalar field acts as a source for scalar operators in the boundary theory. One should hence associate the bulk action with terms in the conformal field theory action of the form

$$I = \int d\lambda d\Delta d\sigma u^{-2+\frac{2}{3}\Delta-\frac{2}{3}\lambda} \Phi(\lambda, x) \mathcal{O}_\Delta(x), \quad (4.23)$$

where  $-\lambda$  is the conformal weight of the boundary scalar field data and  $\Delta$  is the conformal weight of the operator  $\mathcal{O}$ . The  $u$  dependence of this action is determined by the requirement of conformal invariance: suppose that the scalar field data behaves as

$$\Phi(u, x) \sim u^{-\lambda} \Phi^b(x), \quad (4.24)$$

on the boundary. That is, the data scales with  $u$  which is a positive function that has a simple zero on the boundary. Following the same arguments as in [3], the definition of  $\Phi^b(x)$  depends on our particular choice of function, and if we transform  $u \rightarrow e^w u$  then  $\Phi^b(x) \rightarrow e^{w\lambda} \Phi^b(x)$ . Under the same transformation the measure transforms as  $d\sigma \rightarrow e^{2w} d\sigma$ , since the measure is of conformal weight  $-3$ . The degeneracy of the boundary implies that an operator of conformal weight  $\Delta$  scales as  $e^{-2w\Delta/3}$  and since the action must be conformally invariant this implies that it must be of the form (4.23).

Comparing the forms of (4.23) and (4.22) we see that we must identify

$$\begin{aligned} \mathcal{O}_\Delta(x) = \int d\bar{x}_1 d\bar{x}_2 & \left[ \left( \frac{2}{3} \Delta \right) K^{(m)}(\alpha_1, \beta_1) \Phi^{(m)}(\Delta - 3) \right. \\ & \left. - (2\alpha_2 K^{(m)}(\alpha_2 + 1, \beta_2) \cos \theta + 2\beta_2 K^{(m)}(\alpha_2, \beta_2 + 1) \sin \theta) \Phi^{(m)}(\Delta - 6) \right], \end{aligned} \quad (4.25)$$

where

$$\alpha_1 + \beta_1 = \frac{2\Delta}{3}; \quad \alpha_2 + \beta_2 = \frac{2\Delta}{3} - 2. \quad (4.26)$$

In addition  $\alpha_i, \beta_i$  satisfy the constraint (4.19). This gives us the expectation value of the operator and functionally differentiating this expression again will give us the two-point functions. Writing the boundary value of the scalar field as the mode expansion (4.17), then this expression may be inverted as

$$\Phi^{(m)}(\lambda, x_1, x_2) = u^{\frac{2\lambda}{3}} \int \frac{d\theta}{\sin^2 \theta \cos^2 \theta} Y^{(m)}(\lambda, \theta) \Phi^m(u, x), \quad (4.27)$$

where we are again ignoring  $\lambda$  dependent normalisation factors; in fact, to get the normalisation factors right, we would have to say more carefully how we are going to regularise these

formally divergent integrals. The two-point function  $\langle \mathcal{O}_\Delta(x) \mathcal{O}_{\Delta'}(y) \rangle$  is given by functionally differentiating (4.25) with respect to

$$u^{-2+\frac{2\Delta'}{3}} \Phi^{(m)}(u, x). \quad (4.28)$$

Let us consider the result of differentiating the first term in (4.25); then the correlation function is only non-zero for  $\Delta = \Delta'$  and we get a two-point function

$$\langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(\bar{x}) \rangle = \frac{2\Delta \cos^{\alpha_1} \theta \cos^{\alpha_1} \bar{\theta} \sin^{\beta_1} \theta \sin^{\beta_1} \bar{\theta}}{3 (\Delta x_1)^{2\alpha_1} (\Delta x_2)^{2\beta_1}}. \quad (4.29)$$

(4.29) gives the leading order behaviour of the correlation function for operators of the same conformal weight as  $u \rightarrow 0$ . Subleading behaviour is derived from the last two terms in (4.25)

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(\bar{x}) \rangle = & -u^2 \left\{ 2\alpha_2 \frac{\cos^{\alpha_2+2} \theta \cos^{\alpha_2} \bar{\theta} \sin^{\beta_2} \theta \sin^{\beta_2} \bar{\theta}}{(\Delta x_1)^{2(\alpha_2+1)} (\Delta x_2)^{2\beta_2}} \right. \\ & \left. + 2\beta_2 \frac{\cos^{\alpha_2} \theta \cos^{\alpha_2} \bar{\theta} \sin^{\beta_2+2} \theta \sin^{\beta_2} \bar{\theta}}{(\Delta x_1)^{2\alpha_2} (\Delta x_2)^{2(\beta_2+1)}} \right\}. \end{aligned} \quad (4.30)$$

However, differentiating the last two terms in (4.25) also gives a non-zero leading order contribution to the correlation function

$$\begin{aligned} \langle \mathcal{O}_\Delta(x) \mathcal{O}_{\Delta-3}(\bar{x}) \rangle = & - \left\{ 2\alpha_2 \frac{\cos^{\alpha_2+2} \theta \cos^{\alpha_2} \bar{\theta} \sin^{\beta_2} \theta \sin^{\beta_2} \bar{\theta}}{(\Delta x_1)^{2(\alpha_2+1)} (\Delta x_2)^{2\beta_2}} \right. \\ & \left. + 2\beta_2 \frac{\cos^{\alpha_2} \theta \cos^{\alpha_2} \bar{\theta} \sin^{\beta_2+2} \theta \sin^{\beta_2} \bar{\theta}}{(\Delta x_1)^{2\alpha_2} (\Delta x_2)^{2(\beta_2+1)}} \right\}. \end{aligned} \quad (4.31)$$

That is, operators of unequal conformal weight have a non-vanishing two point function! Now (4.29) and (4.31) have precisely the same form as terms in the integral (4.5); one can check that the equivalent between indices in the two expressions. Thus we have explicitly verified that the scalar two point functions are reproduced from the action for bulk scalar fields.

For completeness, let us now sketch the principle features of the correspondence for the coset space  $SO(3,1)/SO(2)$  given in (2.13). The isometries of this space are generated by the  $sl(2, R) \times sl(2, R) \times so(2)$  algebra

$$l_0^i = i\partial_{\phi_i}; \quad l_{\pm 1}^i = ie^{\pm\phi_i} (\coth \rho_i \partial_{\phi_i} \mp i - \partial_{\rho_i}); \quad k = i\partial_\psi. \quad (4.32)$$

One can find primary fields satisfying

$$l_0^i \Psi = h^i \Psi; \quad k \Psi = h_k \Psi; \quad l_1^i \Psi = 0, \quad (4.33)$$

such that the bulk mass is related to these conformal weights as

$$m^2 = h_i(h_i + 1) + h_k^2. \quad (4.34)$$

By analogy to the above analysis for  $H^2 \times H^2$ , if one considers the correlation function of two operators of conformal weights  $\Delta_1$  and  $\Delta_2$  on the hypersurface  $u \rightarrow 0$  defined from (2.15), then

$$\langle \mathcal{O}_{\Delta_1}(x) \mathcal{O}_{\Delta_2}(\bar{x}) \rangle = \sum_{n,l} f_{n,l}(\Delta_1, \Delta_2) e^{in\Delta\psi} U_l(\theta, \bar{\theta}, \Delta\phi_i, \Delta_1, \Delta_2), \quad (4.35)$$

where the coefficients  $f_{n,l}$  are not fixed by the conformal symmetry but conformal covariance requires that  $U_l$  satisfies four equations of the form

$$\begin{aligned} i(\partial_{\phi_1/\phi_2} - i \sin \theta \cos \theta \partial_{\bar{\theta}} + e^{i\Delta\phi_1/(-\Delta\phi_2)}(\partial_{\bar{\phi}_1/\bar{\phi}_2} - i \sin \bar{\theta} \cos \bar{\theta} \partial_{\bar{\theta}})U_l \\ = -\frac{2}{3}(\Delta_1 \sin^2 \theta + \Delta_2 \sin^2 \bar{\theta} e^{i\Delta\phi_1/(-\Delta\phi_2)})U_l; \\ i(\partial_{\phi_1/\phi_2} + i \sin \theta \cos \theta \partial_{\bar{\theta}} + e^{(-i\Delta\phi_1)/\Delta\phi_2}(\partial_{\bar{\phi}_1/\bar{\phi}_2} + i \sin \bar{\theta} \cos \bar{\theta} \partial_{\bar{\theta}})U_l \\ = -\frac{2}{3}(\Delta_1 \cos^2 \theta + \Delta_2 \cos^2 \bar{\theta} e^{(-i\Delta\phi_1)/\Delta\phi_2})U_l. \end{aligned} \quad (4.36)$$

Terms in the summation (4.35) should then be reproduced by considering the action for bulk massive scalar fields.

It is interesting to note that although the bulk scalar field action for  $H^2 \times H^2$  contains terms which diverge as  $u \rightarrow 0$  the two-point functions are actually regular in this limit. The same behaviour was found for the Bergman metric; in fact in this case the action for a massive scalar is independent of the IR regularisation parameter [10]. Thus, we don't need to keep  $u$  finite in the correlation functions.

If we keep  $u$  finite in the boundary field theory, the two point functions will still be fixed by the conformal symmetry group of a constant  $u$  hypersurface. To compare with the bulk theory, we should introduce propagators describing sources at finite  $u$  and repeat the above analysis. It would be interesting to consider the flow of the propagators as one changes  $u$ , particularly in the context of making more precise the correspondence at finite  $u$ . One could also compare the spectrum on  $H^2 \times H^2$  with operators appearing in the boundary theory.

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## APPENDIX: EFFECTIVE ACTIONS ON THE SQUASHED THREE SPHERE

Eigenvalues of the scalar operator (3.32) used in [32] are

$$\lambda = \frac{1}{4l_3^2}(n^2 + 4(l_3^2 - 1)(q + \frac{1}{2})(n - q - \frac{1}{2})), \quad (A1)$$

with degeneracy  $n = 1, \infty$ . We have set  $l_3^2 = l^2 R^2$  and  $q$  runs from 0 to  $(n - 1)$ . For our conformally coupled operator (3.33) we just need to shift the eigenvalues, and can hence write the partition function as  $W_{sc} = \frac{1}{2}\zeta'(0)$  where

$$\zeta(s) = (2l_3)^{2s} \sum_{n=1}^{\infty} \sum_{q=0}^{n-1} \frac{n}{(n^2 + 4(l_3^2 - 1)(q + \frac{1}{2})(n - q - \frac{1}{2}) - \frac{l_3^4}{4})^s}. \quad (\text{A2})$$

The approach of [32] was to apply the Plana summation formula to the  $q$  summation; in the extreme oblate limit  $l_3 \rightarrow \infty$  it is then quite easy to find the dominant term in the zeta function which behaves as  $l_3^4$ .

Following the same approach here, the key point is that, although sub-dominant terms in the Plana summation formula are affected by the shift in eigenvalues, the dominant term in the extreme prolate limit is determined by a very similar term to that in [32]

$$\zeta(s) \approx 2i(2l_3)^{2s} \int_0^{\infty} \frac{dt}{\exp(2\pi t) + 1} \left\{ \frac{n}{(n^2 + 4(l_3^2 - 1)(t^2 - itn) - \frac{l_3^4}{4})^s} - (t \rightarrow -t) \right\}, \quad (\text{A3})$$

which we can analyse by the Watson-Sommerfeld method to give a leading order contribution of

$$W_{\text{sc}} \approx \frac{3l_3^4}{2\pi^2} \zeta_R(3), \quad (\text{A4})$$

as was found in [32].

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